Mid-Semestral Exam
Algebra-I
B. Math - First year
2016-2017

Time: 3 hrs
Max score: 100

Answer **Question 1** and any **4** from the rest.

(1) State true or false. Justify your answers. No marks will be awarded in the absence of proper justification.
(a) Suppose a group G contains exactly eight elements of order 10. Then G has 3 cyclic subgroups of order 10.
(b) There are 90 elements of order 4 in S_6.
(c) If G is a group having exactly one nontrivial proper subgroup, then G is cyclic, of order p^2 for some prime p.
(d) There exist infinite groups in which every element has finite order.
(e) The set $\mathbb{R}_{>0}$ of all positive reals is the only subgroup of index 2 in the multiplicative group of nonzero reals $\mathbb{R}^\times$. (4 × 5)

(2) Let G be a cyclic group of order n.
(a) Show that every subgroup of G is cyclic.
(b) Show that for each k dividing n, there exists a unique subgroup of order k in G, and
(c) Show that for any divisor d of n, G contains exactly $\phi(d)$ elements of order d. Deduce the formula $\sum_{d/n} \phi(d) = n$. (4+8+8)

(3) (a) If K is a subgroup of G and N is a normal subgroup of G, prove that
(i) $KN = \{xy \in G \ / x \in K, y \in N\}$ is a subgroup of G,
(ii) $K \cap N$ is a normal subgroup of K, and
(iii) KN/N is isomorphic to $K/(K \cap N)$.
(b) If M and N are normal subgroups of of G and $N \subseteq M$, prove that $(G/N)/(M/N) \cong G/M$. (12+8)

(4) (a) Define conjugation action of a group G on itself.
(b) Show that the number of distinct conjugates of an element $g \in G$ is the index of the centraliser $C_G(g)$ in G.
(c) Establish the Class Equation. (2+8+10)

(5) (i) State and prove Cauchy's theorem for finite abelian groups.
(ii) Using Class Equation, or otherwise, prove Cauchy's theorem for finite non-abelian groups. (10+10)

(6) (a) Show that $C_{S_n}((12)(34)) = 8 \times (n-4)!$ for all $n \geq 4$. Determine the elements of the centraliser explicitly.
(b) Show that if n is odd, the set of all n-cycles consists of two conjugacy classes of equal size in A_n. (8+12)